A New Basis for k-Local Class Functions on $\mathfrak{S}_n$

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Advised by Professor Michael Orrison

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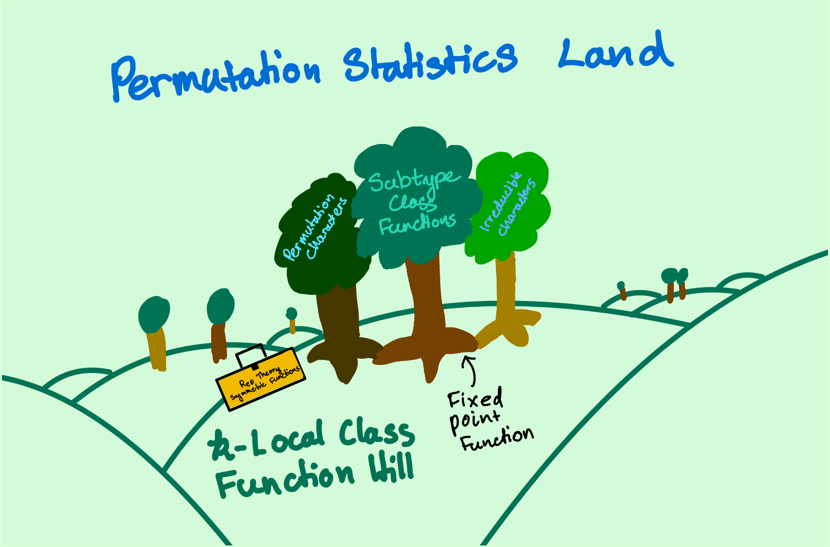
Permutation Statistics Land
Permutation Characters
Subtype Class Functions
Irreducible Characters
Rep Theory
Symmetric Functions
Fixed point Function
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The Symmetric Group

Definition

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The *cycle type* of a permutation is the list of lengths of the cycles of the permutation when written in cycle notation.

Example

The cycle type of (1 4 2)(3)(5 6) is $(3, 2, 1)$. The cycle type of (1)(2)(3)(4)(5 6) is $(2, 1^4)$.

Fixed Point Function

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Define the fixed point function by $\mathrm{Fix}(\sigma) = \#\{i \in [n] \mid \sigma(i) = i\}$.

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Observation

The number of fixed point is the same as the number of 1-cycles in the cycle decomposition of the permutation...

Class Functions

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Example

The fixed point function is a class function. For instance, all permutations of cycle type $(2, 1^4)$ have 4 cycles of length 1.

k-Local Functions

A function is k-local if it can be evaluated by only looking at k numbers in the permutation at a time!

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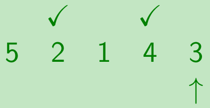


Examples

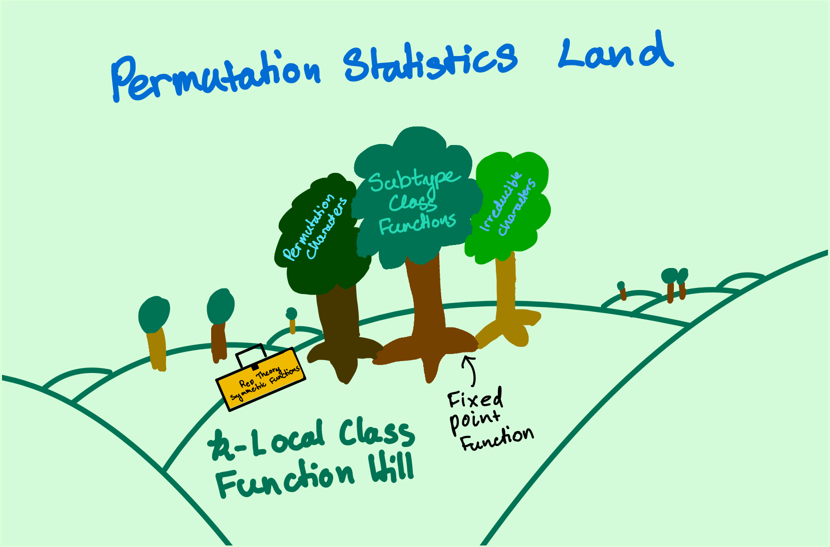
Example

Most permutation statistics are k-local, including

- Fixed Points
- Excedances
- Inversions
- Major Index
- k-cycle counts
- Pattern containment

but not

- Longest Increasing Subsequence
- Number of Cycles.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The space of k-local class functions...

- includes functions we care about like the fixed point function, irreducible characters, and permutation characters!
- is the intersection of two types of "nice" functions: k-local functions and class functions!
- we can use our toolbox of representation theory and symmetric functions!

Irreducible Characters

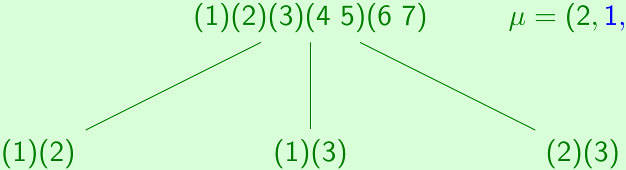
Proposition

The subset of irreducible characters of $\mathfrak{S}_n$, $\{\chi^\lambda \mid \lambda_1 \geq n-k\}$, forms a basis for the space of k-local class functions.

Subtype Class Functions

$$(1)(2)(3)(4\ 5)(6\ 7) \qquad \mu = (2, 1, 1)$$

Subtype Class Functions



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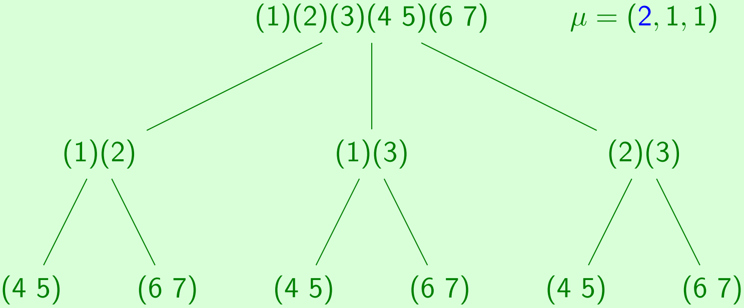

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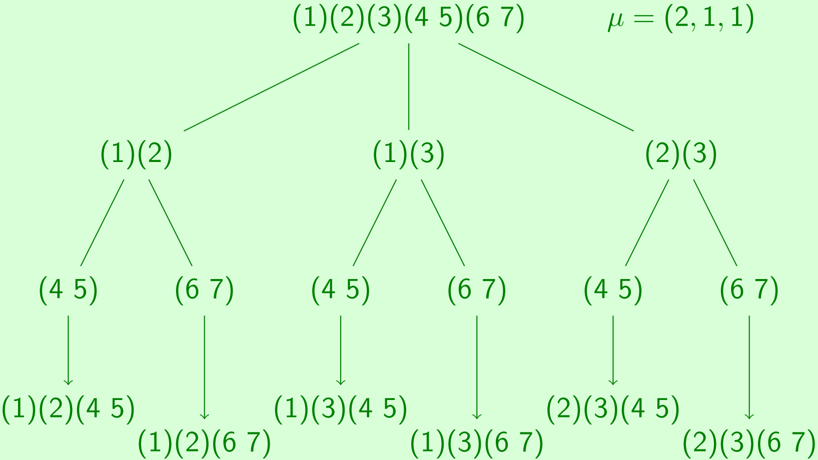

Subtype Class Functions

Definition

Suppose $\mu = 1^{m_1}2^{m_2}\ldots$ and $m_1 + 2m_2 + 3m_3 + \ldots = k$. The subtype class function of type μ, denoted by T^μ, is the number of ways to choose m_1 1-cycles, m_2 2-cycles, etc from the permutation.

$$T^\mu(\sigma) = \prod_{j=1}^{k} \binom{T^{(j)}(\sigma)}{m_j}$$

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Proposition (Macdonald 1995)

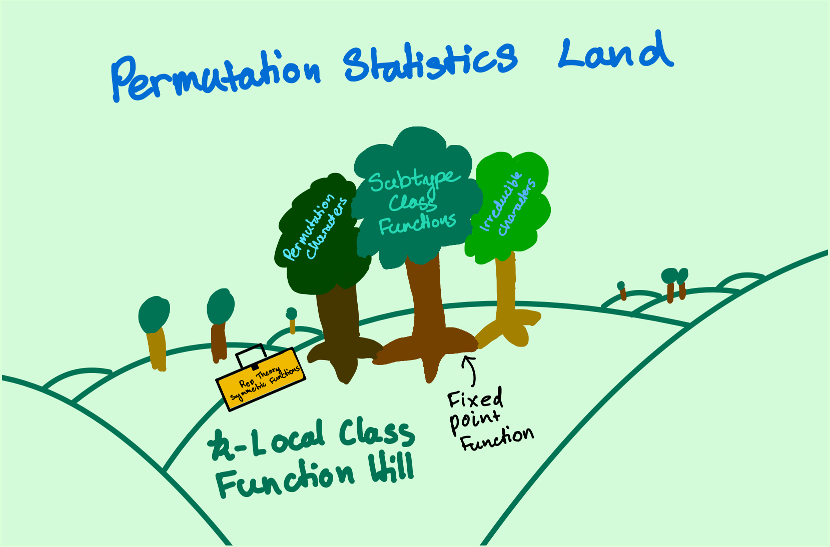
If σ has cycle type λ and $\lambda = \mu \cup \rho$, then

$$T^\mu(\sigma) = \frac{z_\lambda}{z_\rho z_\mu}.$$

Subtype Class Functions

Proposition

The subset of subtype class functions, $\{T^{\mu} \mid \mu \vdash k, \mu_1 \leq n-k\}$, forms a basis for the space of k-local class functions.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References

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